Canonical Forms of Self-adjoint Operators and Associated Stress-Energy Tensors. *

F. Sánchez Martín † ‡

October 15, 2006

Abstract

An analysis of selfadjoint operators is presented in the frame of lorentzian space-time of dimension 4, L_4 with signature (-1,1,1,1). The type of canonical operators and stress-energy tensor associated are deduced for L_4.

This analysis is aimed to the physical interpretation of results gotten in classification of stress-energy tensor associated. The classification implemented is not thorough. It is mainly devoted to analysis and study of most important stress energy tensors, treating them going into important details.

Results agree classic. **Classification presented by Hall G S [1] and others is reinterpreted** .

It is found out (in stress-energy tensor type 2, associated to radical space with index 2) that a flux (by example caloric flow) inherent to a reference system at rest, involves a term similar to radiation scheme too.

The caloric flux must be in the stress-energy scheme type 2.

---

*With an analysis and study of canonical forms of selfadjoint operators, associated stress-energy tensors are shown in the context of General Theory of Relativity. Some interpretations are also shown.

†Licenciate in Physical Sciences (Universidad Complutense de Madrid; Spain)

‡Public Upper Body of Systems and Technologies in Computers and Communications (Spain). Public Upper Body of Meteorology (Spain)

1 Introduction

Canonical forms of selfadjoint operators on lorentzian spaces of 4 and 5 dimensions have been dealed with a wider context in General Theory of Relativity, by differents authors; by example see [3].

I am geting the invariant subspaces study step by step from most simple cases, that is from simple eigenvectors and eigenvalues [1] different or multiple (usually known as eigenvectors or also radical spaces of index 1, and eigevalues) until subspaces corresponding to irreducible minimum polynomial of degree 2 and radical subspaces of index 2 and 3.

Classically many people dealt with the simplest cases (4 eigenvalues simple different or multiple), which allowed have a vision similar to the eucledian spaces, thus facilitating the physical interpretation of stress-energy tensor. However important differences exist between the eucledian and lorentzian spaces arising from the fact that lorentzian spaces can admit radical subspaces of higher index to 1, and minimum polynomials irreducible of degree 2, besides spaces radicals of index 1 (simple eigenvalues), as in the eucledian spaces. Consequently appear several types of stress-energy tensors not covered in many of the classic works of General Relativity (see [2]). Some of them are ruled out by not fulfill the strong or weak conditions from stress-energy tensor.

The authors who have studied the classification of selfadjoint operators and the associated stress-energy tensors , have used the classification of Segré in large measure, spreading to lorentzian spaces of more than 4 dimensions [3].

In this work, we proceed only in the body of the real numbers unless otherwise specified, and in the lorentzian spaces of dimension 4 and signature (- 1.1,1,1).

This article is part of a group of articles oriented to analysis and interpretations of stress-energy tensor in General Theory of Relativity and other topics.

[1] When we speak of simple eigenvectors and eigenvalues we are concerned to invariant subspaces of dimension 1, or to radical subspaces of index 1. It is the simplest case to study.

1.1 Notations and symbols

L_n symbolize a n-dimensional lorentzian space , of signature (-1,1,1,...,1) ; $n \leq 4$
I symbolizes null-cone.
I_1 symbolizes a null straight line.
E_n symbolizes an n-dimensional euclidean space.

For eigenvalues and eigenvectors we use the expression ***simples*** , meaning radical spaces of index 1.

Vectors are shown with arrows.

Tensors are shown with boldtype cap letters.

The matrix reprsentation of a operator, tensor, or other mathematical object wear paretheses ; for example **(T)**, is the matrix representation of tensor **T**.

The scalar product of two vectors $\vec{X}$ and $\vec{Y}$ is shown by $\mathbf{g}(\vec{X}, \vec{Y})$ where **g** is the metric tensor. It also is shown by $\vec{X}.\vec{Y}$

2 Simple eigenvalues

It is the case corresponding to radical subspaces of index 1.

A selfadjoint operator **M** on a lorentzian space L_4 verifies

$$\mathbf{g}(\mathbf{M}(\vec{X}), \vec{Y}) = \mathbf{g}(\vec{X}, \mathbf{M}(\vec{Y})) \tag{1}$$

where $\vec{X}$ e $\vec{Y}$ are vectors of lorentzian space L_4, that is $\vec{X}, \vec{Y} \in L_4$

Minimum polinomials of these invariant subspaces have the form :

$$P(\mathbf{M}) \equiv \mathbf{M} - \lambda_i \mathbf{I}; i = 0, \ldots, 3 \tag{2}$$

2.1 Simple and distinct eigenvalues

We have:

$$P(\lambda_i) \equiv \mathbf{M} - \lambda_i I; \lambda_i \neq \lambda_j; i, j = 0, \ldots, 3 \tag{3}$$

Therefore we have 4 eignvalues λ_i ; i=0,..,3 and 4 eigenvectors $\overrightarrow{X_i}$, i=0,..,3 that verifie

$$\mathbf{M}(\vec{\mathbf{X}}_i) = \lambda_i \vec{\mathbf{X}}_i \tag{4}$$

Then we attend to see specifically, the relevant cases of null and nonull eigenvectors.

2.1.1 Nonull eigenvectors with different eigenvalues

In this case eigenvectors are ortogonal.

$$\mathbf{g}(\mathbf{M}(\vec{\mathbf{X}}_i), \vec{\mathbf{X}}_j) = \mathbf{g}(\vec{\mathbf{X}}_i, \mathbf{M}(\vec{\mathbf{X}}_j))$$

that is

$$\mathbf{g}(\vec{\mathbf{X}}_i, \vec{\mathbf{X}}_j)(\lambda_i - \lambda_j) = 0; i \neq j \tag{5}$$

As eigenvalues are differents, that is $\lambda_i \neq \lambda_j$, we have $\mathbf{g}(\vec{\mathbf{X}}_i, \vec{\mathbf{X}}_j) = 0$ and therefore the eigenvectors are orthogonal

2.1.2 Null eigenvectors.

The null vectors point out a fundamental difference between the eucledian and lorentzian spaces. It is always necessary to devote them special attention.

The lorentzian space or subspace (on which operator is implemented), is null in case of one null eigenvector and remaining eigenvectors nonull, all of them with different eigenvalues, .

The demonstration is obvious.

In case of two or more null eigenvectors, within the context of point 2.1., these would be orthogonal, therefore they constitute one eigenvector.

The eigenvalues of these null eigenvectors must be the same. Really this part of the paragraph must be incorporate to the next.

2.2 Simple multiple eigenvectors.

In case of multiple simple eigenvalues in space or lorentzian subspaces , several values λ_i are equal.

We write $\lambda_i = \lambda$ for eigenvalues that are equal.

2.2.1 Simple multiple eigenvectors relationship .

In these cases $\lambda_i = \lambda_j = \lambda$, $i, j \leq n$, $0 \leq n \leq 3$.
According (5) , it is not necessary $\mathbf{g}(\vec{X}_i, \vec{X}_j) = 0$.

That is, ***in this case is not necessary that eigenvectors are orthogonal.***

Let $\vec{V} = \sum a_i \vec{X}_i$ remaining $\overrightarrow{X}_i$ multiple eigenvectors. Then

$$\mathbf{M}(\vec{V}) = \mathbf{M}(\sum a_i \vec{\mathbf{X}}_\mathbf{i}) = \lambda \sum a_i \vec{\mathbf{X}}_\mathbf{i} = \lambda \vec{\mathbf{V}}$$

Therefore ***all vectors of space generated by the eigenvectors are simple multiple eigenvectors.***

If one of the eigenvectors is temporal, space or subspace generated is lorentzian. It contains at least 2 null eigenvectors (is assumed that the dimension of this space or subspace is n, $1 < n \leq 4$.

2.2.2 Null directions in case of simple multiple vectors which include temporal vectors.

In this case all the vectors are simple multiple, and because of the temporal subspace be generated by them, there are particularly two or more null vectors.
In some cases each of these null vectors is orthogonal to a subspace of 2 dimensions.
This happens in the case of operator associated with the electromagnetic field where eigenvalues are $(-\lambda, -\lambda, \lambda, \lambda)$. In this case we have a temporal bidimensiona lorentzian subspace L_2 and 2 null eigenvectors.

2.3 Stress-energy tensor class 1; Segré [1,111] .

A symmetrical operator is associated with an symmetric tensor. In the case of simple eigenvalues we choose a basis of orthogonal eigenvectors (if the eigenvalues are simple and distinct, and not zero, they are orthogonal; in the case of equal eigenvalues we simply an orthogonal reference). The eigenvalues of the symmetrical operator $\mathbf{M}$, are $(-\rho, p_1, p_2, p_3,)$.

The symetric operator $\mathbf{M}$, and the metric tensor $\mathbf{G}$, are represented by the matrices:

$$\mathbf{(M)} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p_1 & 0 & 0 \\ 0 & 0 & p_2 & 0 \\ 0 & 0 & 0 & p_3 \end{pmatrix} ; \mathbf{(G)} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The associated matrix to tensor $\mathbf{T}$ is :
$(\mathbf{T}) = \mathbf{(M)(G)}$ or $(\mathbf{T}) = \mathbf{(G)(M)}$ depending on whether components covariants or contravariants on the elements of $\mathbf{(G)}$.
That is to say:

$$\mathbf{(T)} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p_1 & 0 & 0 \\ 0 & 0 & p_2 & 0 \\ 0 & 0 & 0 & p_3 \end{pmatrix}$$

Using the tetradic formulation would:

$$\mathbf{T} = \rho\vec{\mathbf{U}}\otimes\vec{\mathbf{U}} + p_1\vec{\mathbf{X}}_1\otimes\vec{\mathbf{X}}_1 + p_2\vec{\mathbf{X}}_2\otimes\vec{\mathbf{X}}_2 + p_3\vec{\mathbf{X}}_3\otimes\vec{\mathbf{X}}_3 \qquad (6)$$

As a null real reference (also called pseudoortonormal) we use a base with two null vectors $\mathbf{n}$ and $\mathbf{l}$ and the other ortonormal (making a change of basis; see Annex A); would:

$$\mathbf{T} = \tfrac{1}{4}((p_1+\rho)(\tfrac{1}{\alpha^2}\vec{\mathbf{l}}\otimes\vec{\mathbf{l}} + \tfrac{1}{\beta^2}\vec{\mathbf{n}}\otimes\vec{\mathbf{n}})$$
$$+\tfrac{1}{\alpha\beta}(p_1-\rho)(\vec{\mathbf{l}}\otimes\vec{\mathbf{n}} + \vec{\mathbf{n}}\otimes\vec{\mathbf{l}})) + p_2\vec{\mathbf{X}}_2\otimes\vec{\mathbf{X}}_2 + p_3\vec{\mathbf{X}}_3\otimes\vec{\mathbf{X}}_3$$

If we think over the restriction to a tetradic frame then $\alpha = \beta = \frac{\sqrt{2}}{2}$ (see Annex B); we have

$$\mathbf{T} = \tfrac{1}{2}(p_1+\rho)(\vec{\mathbf{l}}\otimes\vec{\mathbf{l}} + \vec{\mathbf{n}}\otimes\vec{\mathbf{n}}) + \tfrac{1}{2}(p_1-\rho)(\vec{\mathbf{l}}\otimes\vec{\mathbf{n}} + \vec{\mathbf{n}}\otimes\vec{\mathbf{l}})$$
$$+p_2\mathbf{X}_2\otimes\mathbf{X}_2 + p_3\mathbf{X}_3\otimes\mathbf{X}_3$$

This formula is equal to that obtained by other authors especially G. S. Hall and D. A. Nesgm ([1]). This is the tensor that is in the classic works of the general relativity ([2]). Includes perfect fluid, electromagnetic field, etc. ([2]).

3 Radical polinomial of index 2; type Segré [2, 11] and degenerancies.

In this case we have a minimum polynomial

$$\mathbf{P}_2(\mathbf{M}) \equiv (\mathbf{M} - \lambda\mathbf{I})^2$$

on a lorentzian space L_2 verifying

$$\mathbf{P}_2(\vec{\mathbf{X}}) \equiv (\mathbf{M} - \lambda\mathbf{I})^2\vec{\mathbf{X}} = \vec{\mathbf{0}}$$

for every $\vec{\mathbf{X}} \in L_2$.

In the radical polynomial index 2 is verified

$$(\mathbf{M} - \lambda\mathbf{I}) \neq (\mathbf{0}) \ ; \ (\mathbf{M} - \lambda\mathbf{I})^2 = (\mathbf{0})$$

We construct the operator also symetric: $\mathbf{H} = (\mathbf{M} - \lambda\mathbf{I})$.
For every $\vec{\mathbf{X}} \in L_2$ is verified

$$\begin{aligned} \mathbf{H}(\vec{\mathbf{X}}) &= \vec{\mathbf{Y}} \\ \mathbf{H}(\vec{\mathbf{Y}}) &= \vec{\mathbf{0}} \end{aligned} \tag{7}$$

where $\vec{\mathbf{Y}}$ proves to be an eigenvector itself.

3.1 $\vec{\mathbf{Y}}$ is a null eigenvector.

In fact:
As $\mathbf{H}$ is selfadjoint.

$$\vec{\mathbf{X}}\mathbf{H}(\vec{\mathbf{Y}}) = \vec{\mathbf{Y}}\mathbf{H}(\vec{\mathbf{X}})$$

and in according to 7 we have

$$\vec{\mathbf{Y}}\mathbf{H}(\vec{\mathbf{X}}) = \vec{\mathbf{Y}}^2 = \vec{\mathbf{X}}\mathbf{H}(\vec{\mathbf{Y}}) = 0$$

Therefore $\vec{\mathbf{Y}}$ is a null vector besides being an eigenvector itself.

3.2 Matrix representation of operators respect a null base, (pseudo-ortonormal).

In the lorentzian space $\mathbf{L}_2$ generated by vectors $\vec{\mathbf{X}}$ e $\vec{\mathbf{Y}}$, we can choose a base constituted by 2 null vectors in which $\vec{\mathbf{X}}$ is a null vector but is not an eigenvector. With the aim to use the current nomenclature of null vectors, we make $\vec{\mathbf{X}} \equiv \vec{\mathbf{l}}$ and $\vec{\mathbf{Y}} \equiv \vec{\mathbf{n}}$.
$\vec{\mathbf{l}}$ and $\vec{\mathbf{n}}$ are unique null vectors defined by the operator.

The matrix representation of $\mathbf{H}$ on the basis $\{\vec{\mathbf{l}}, \vec{\mathbf{n}}\}$ is
$(\mathbf{H}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
and the matrix representation of the operator $\mathbf{M}$ is:

$$(\mathbf{M}) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

Once defined the metric, $\mathbf{G}$ is

$$w = \mathbf{G}(\vec{\mathbf{l}}, \vec{\mathbf{n}})$$

That is the scalar product of $\vec{\mathbf{l}}.\vec{\mathbf{n}}$.
The metric tensor in $\mathbf{L}_2$ has the form ,with matrix covariant components:

$$(\mathbf{G}) = \begin{pmatrix} 0 & w \\ w & 0 \end{pmatrix}$$

and with contravariant components:

$$(\mathbf{G}) = \begin{pmatrix} 0 & 1/w \\ 1/w & 0 \end{pmatrix}$$

3.3 Orthogonality of null eigenvector regard the remaining invariant subspaces index 1.

The complementary invariant subspace E_2 to lorentzian subspace L_2 which we are concerned, is euclidean. Let $\vec{\mathbf{X}}_i$ and λ_i , $i = 2, 3$ eigenvectors and eigenvalues, at $E2$.
For L_2 we keep the same nomenclature.
For $i = 2, 3$ we have

$$\vec{\mathbf{n}}.\mathbf{H}(\vec{\mathbf{X}}_\mathbf{i}) = \lambda_i \vec{\mathbf{X}}_\mathbf{i}.\vec{\mathbf{n}} \text{ and } \vec{\mathbf{n}}.\mathbf{H}(\vec{\mathbf{X}}_\mathbf{i}) = \vec{\mathbf{X}}_\mathbf{i}.\mathbf{H}(\vec{\mathbf{n}}) = 0$$

,theerefore $\vec{\mathbf{n}}.\vec{\mathbf{X}}_i = 0$ is derived if $\lambda_i \neq 0$.
And hence ***the null eigenvector $\vec{n}$ of lorentzian space L_2 is ortogonal to eucledian space complementary to L_2.***

3.4 Tensor form associated to the operator.

The tensor $\mathbf{T}$ associated to $\mathbf{M}$ is ,using the matrix representation of their covariant components

$$\mathbf{T} = (\mathbf{M})(\mathbf{G}) = \begin{pmatrix} w & \lambda w \\ \lambda w & 0 \end{pmatrix} \tag{8}$$

And in the matrix representation of its contravariants components

$$\mathbf{T} = (\mathbf{G})(\mathbf{M}) = \begin{pmatrix} 1/w & \lambda/w \\ \lambda/w & 0 \end{pmatrix} \tag{9}$$

We call **stress-energy tensor of type 2** to stress-energy tensor associated with a radical operator index 2.

3.4.1 Stress-energy tensor of type 2 respect a null base (pseudo-ortonormal).

Let a base in L_4 $\vec{\mathbf{l}}, \vec{\mathbf{n}}, \vec{\mathbf{X}}_{\mathbf{2}}, \vec{\mathbf{X}}_{\mathbf{3}}$. On this null base the stress energy tensor is :

$$\mathbf{T} = \frac{1}{w}(\vec{\mathbf{n}} \otimes \vec{\mathbf{n}} + \lambda(\vec{\mathbf{l}} \otimes \vec{\mathbf{n}} + \vec{\mathbf{n}} \otimes \vec{\mathbf{l}})) + p_2 \vec{\mathbf{X}}_{\mathbf{2}} \otimes \vec{\mathbf{X}}_{\mathbf{2}} + p_3 \vec{\mathbf{X}}_{\mathbf{3}} \otimes \vec{\mathbf{X}}_{\mathbf{3}} \tag{10}$$

The equation (10) is similar to that obtained by other authors (see [1]).

3.4.2 Stress-energy tensor type 2 respect an ortonormal base

Using the tetradic notation, let us call $\mathbf{U}, \mathbf{X_1}, \mathbf{X_2}$, and $\mathbf{X_3}$ vectors of the base orthonormal in L_4. Turning to a reference orthonormal system and representing $\vec{\mathbf{l}}$ and $\vec{\mathbf{n}}$ as shown in *annex A*, we have:

$\mathbf{T} = (\frac{\beta^2}{w} + \lambda)\vec{\mathbf{X}}_{\mathbf{1}} \otimes \vec{\mathbf{X}}_{\mathbf{1}} + (\frac{\beta^2}{w} - \lambda)\vec{\mathbf{U}} \otimes \vec{\mathbf{U}} + \epsilon\frac{\beta^2}{w}(\vec{\mathbf{X}}_{\mathbf{1}} \otimes \vec{\mathbf{U}} + \vec{\mathbf{U}} \otimes \vec{\mathbf{X}}_{\mathbf{1}}) + p_2 \vec{\mathbf{X}}_{\mathbf{2}} \otimes \vec{\mathbf{X}}_{\mathbf{2}} + p_3 \vec{\mathbf{X}}_{\mathbf{3}} \otimes \vec{\mathbf{X}}_{\mathbf{3}}$
where $\epsilon = \pm 1$ and $w = \overrightarrow{l}.\overrightarrow{n} = 2\alpha\beta$.
For an observer at rest would be $\beta = \alpha = \frac{\sqrt{2}}{2}$; w=1 (see Anex B):
$\mathbf{T} = (\frac{1}{2} + \lambda)\vec{\mathbf{X}}_{\mathbf{1}} \otimes \vec{\mathbf{X}}_{\mathbf{1}} + (\frac{1}{2} - \lambda)\vec{\mathbf{U}} \otimes \vec{\mathbf{U}} + \epsilon\frac{1}{2}(\vec{\mathbf{X}}_{\mathbf{1}} \otimes \vec{\mathbf{U}} + \vec{\mathbf{U}} \otimes \vec{\mathbf{X}}_{\mathbf{1}})$
$+ p_2 \vec{\mathbf{X}}_{\mathbf{2}} \otimes \vec{\mathbf{X}}_{\mathbf{2}} + p_3 \vec{\mathbf{X}}_{\mathbf{3}} \otimes \vec{\mathbf{X}}_{\mathbf{3}}$

3.5 Relevant hints.

Among the most interesting aspects it should be noted:
a).-The strong and weak conditions of energy and temporal flow conditions are : $\rho > 0$ $\lambda \leq 0$ and $\frac{\lambda}{4} \leq p_i \leq -\frac{\lambda}{4}$, $i = 2.3$, as ([1]) and ([3]).
b).-The stress-energy tensor type 2 includes the electromagnetic radiation scheme $\lambda = 0$, the flow (caloric $\lambda \neq 0$), etc.... Its representation on an orthonormal basis is what would receive an observer at

relative rest.
The stress-energy tensor scheme of of caloric flow only has representation in stress-energy tensor type 2.
This implies that **a caloric flow has to be associated always to a term similar to the radiation scheme, which would possibly be manifest in limit situations**. The balance between the terms similar to radiation and caloric conductivity is determined by λ. For values of $\lambda = 0$ we have the case of radiation in a vacuum.
c).- The two (type 1 and type 2), are not reducible one another.
d).- The sum of two or more stress-energy tensors may involve to make adjustments in the conditions weaknesses and strong of the energy and flow and even tuning conditions. Can occur compatibility or incompatibility of topological type (see [4]).
e).- In the radical spaces index 2 there is always a null vector $\vec{\mathbf{n}}$ or $\vec{\mathbf{l}}$ (as explained in point 3.1-) with eigenvalue λ. This is orthonormal to vectors $\vec{\mathbf{X}}_{\mathbf{2}}$ and $\vec{\mathbf{X}}_{\mathbf{3}}$.

In points (b), (c) , (d) and (e) appear relevant topics to be treated apart in other articles.

4 Radical spaces index 3 ; type Segré [3,1].

In them the operator $\mathbf{M}$ verifies:

$$\mathbf{P}_3 \equiv (\mathbf{M} - \lambda\mathbf{I})^3\vec{\mathbf{X}} = \vec{\mathbf{0}}$$

in L_3 being $(\mathbf{M} - \lambda\mathbf{I})^2\vec{\mathbf{X}} \neq \vec{\mathbf{0}}$ and $(\mathbf{M} - \lambda\mathbf{I})\vec{\mathbf{X}} \neq \vec{\mathbf{0}}$.

We construct the operator (also selfadjoint)

$$\mathbf{H} = (\mathbf{M} - \lambda\mathbf{I}).$$

We have

$$\mathbf{H} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Then

$$\mathbf{M} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

Thus for all $\vec{\mathbf{X}} \in L_3$, there are vectors $\vec{\mathbf{Y}}$,$\vec{\mathbf{Z}} \in L_3$ verifying:

$$\mathbf{H}(\vec{\mathbf{X}}) = \vec{\mathbf{Y}}$$
$$\mathbf{H}(\vec{\mathbf{Y}}) = \vec{\mathbf{Z}}$$
$$\mathbf{H}(\vec{\mathbf{Z}}) = \vec{\mathbf{0}}$$

4.1 Some properties.

Let the cyclic base $\{\vec{\mathbf{X}}, \vec{\mathbf{Y}}, \vec{\mathbf{Z}}\}$. We describe below some basic properties of the index 3 lorentzian space L_3.

4.1.1 $\vec{\mathbf{Z}}$ is a null eigenvector.

To be $\mathbf{H}$ selfadjoint we have:

$$\vec{\mathbf{Z}}.\vec{\mathbf{Z}} = \vec{\mathbf{Z}}.\mathbf{H}(\vec{\mathbf{Y}}) = \vec{\mathbf{Y}}.\mathbf{H}(\vec{\mathbf{Z}}) = \vec{\mathbf{0}}$$

4.1.2 $\vec{\mathbf{Y}}$ and $\vec{\mathbf{Z}}$ are ortogonals.

In fact:
Likewise, as $\mathbf{H}$ is selfadjoint we have:

$$\vec{\mathbf{Z}}.\vec{\mathbf{Y}} = \vec{\mathbf{Z}}.\mathbf{H}(\vec{\mathbf{X}}) = \vec{\mathbf{X}}.\mathbf{H}(\vec{\mathbf{Z}}) = \vec{\mathbf{0}}$$

4.1.3 It is verified:

$$\vec{\mathbf{Y}}^2 = \vec{\mathbf{X}}.\vec{\mathbf{Z}}$$

In fact:
$\vec{\mathbf{Y}}^2 = \vec{\mathbf{Y}}.\mathbf{H}(\vec{\mathbf{X}}) = \vec{\mathbf{X}}.\mathbf{H}(\vec{\mathbf{Y}}) = \vec{\mathbf{X}}.\vec{\mathbf{Z}}$

4.1.4 $\vec{\mathbf{Y}}$ is a spacelike vector.

In fact: As $\vec{\mathbf{Y}}$ is orthogonal to $\vec{\mathbf{Z}}$, it can only be spacelike or null. The case null is discarded obviously.

4.1.5 Any vector $\vec{X'}$ generated by the cyclic base $\{\vec{\mathbf{X}}, \vec{\mathbf{Y}}, \vec{\mathbf{Z}}\}$ verifies $\mathbf{H}(\vec{\mathbf{X}}\text{'})^3 = \vec{\mathbf{0}}$.

In fact: let

$$\vec{\mathbf{X}}\text{'} = a_X\vec{\mathbf{X}} + a_Y\vec{\mathbf{Y}} + a_Z\vec{\mathbf{Z}}$$

We have

$$\mathbf{H}(\vec{\mathbf{X}}\text{'}) = a_X\vec{\mathbf{Y}} + a_Y\vec{\mathbf{Z}} = \vec{\mathbf{Y}}\text{'}$$

$$\mathbf{H}(\vec{\mathbf{Y}}\text{'}) = a_X\vec{\mathbf{Z}} = \vec{\mathbf{Z}}\text{'}$$
$$\mathbf{H}(\vec{\mathbf{Z}}\text{'}) = \vec{\mathbf{0}}$$

and hence

$$\mathbf{H}(\vec{\mathbf{X}}\text{'}) = \vec{\mathbf{Y}}\text{'}$$
$$\mathbf{H}(\vec{\mathbf{Y}}\text{'}) = \vec{\mathbf{Z}}\text{'}$$
$$\mathbf{H}(\vec{\mathbf{Z}}\text{'}) = \vec{\mathbf{0}}$$

$\vec{\mathbf{X}}$',$\vec{\mathbf{Y}}$' y $\vec{\mathbf{Z}}$' verify the same properties 4.1.1 to 4.1.4 than $\vec{\mathbf{X}}, \vec{\mathbf{Y}}$ y $\vec{\mathbf{Z}}$; hencefor $\vec{\mathbf{Y}}\text{'}.\vec{\mathbf{Z}}\text{'} = \vec{\mathbf{0}}$ that is $\vec{\mathbf{Y}}\text{'}.\mathbf{H}(\vec{\mathbf{Y}}\text{'}) = \vec{\mathbf{0}}$ for all $\vec{\mathbf{Y}}\text{'} = \mathbf{H}(\vec{\mathbf{X}}\text{'})$.

4.1.6

It is verified

$$\mathbf{L}_3 \overset{\mathbf{H}}{\dashrightarrow} \mathbf{L}_2 \overset{\mathbf{H}}{\dashrightarrow} \mathbf{I}_1$$

or

$$\mathbf{L}_3 \supset \mathbf{L}_2 \supset \mathbf{I}_1$$

where $\mathbf{L}_2 = \mathbf{H}(\mathbf{L}_3)$ and $\mathbf{I}_1 = \mathbf{H}(\mathbf{L}_2)$

4.1.7 Metric tensor associated

In the lorentzian subspace L_3 (generated by $(\vec{\mathbf{X}}, \vec{\mathbf{Y}}, \vec{\mathbf{Z}})$ is defined a scalar product. According with this, and with the previous paragraphs

$$\vec{\mathbf{X}}^2 = a$$
$$\vec{\mathbf{Y}}^2 = b$$
$$\vec{\mathbf{Z}}^2 = 0$$
$$\vec{\mathbf{X}}.\vec{\mathbf{Y}} = c$$
$$\vec{\mathbf{X}}.\vec{\mathbf{Z}} = b$$
$$\vec{\mathbf{Y}}.\vec{\mathbf{Z}} = 0$$

Using the base $[\vec{\mathbf{X}}, \vec{\mathbf{Y}}, \vec{\mathbf{Z}}]$, the metric is

$$\mathbf{G} = \begin{pmatrix} a & c & b \\ c & b & 0 \\ b & 0 & 0 \end{pmatrix}$$

in covariant components.In contravariant components we have:

$$G^{-1} = \frac{1}{b}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -\frac{c}{b} \\ 1 & -\frac{c}{b} & (\frac{c^2}{b^2} - \frac{a}{b}) \end{pmatrix}$$

The determinant of $\mathbf{G}$ is $|\mathbf{G}| = -b^3$. As the subspace L_3 is timelike, then $b > 0$.

4.2 Type 3 stress-energy tensor associated

We know that

$$\mathbf{H} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\mathbf{M} = \mathbf{H} + \lambda\mathbf{I} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

The contravariants components of stress-energy tensor in L_3 are the elements of the matrix

$$(\mathbf{T}_3) = \frac{1}{b}\begin{pmatrix} 0 & 0 & \lambda \\ 0 & \lambda & 1 - \frac{c}{b}\lambda \\ \lambda & 1 - \frac{c}{b}\lambda & -\frac{c}{b} + (\frac{c^2}{b^2} - \frac{a}{b}\lambda) \end{pmatrix}$$

Making $\vec{\mathbf{X}} = \vec{\mathbf{l}}$, $\vec{\mathbf{Y}} = \vec{\mathbf{X}}_2$ and $\vec{\mathbf{Z}} = \vec{\mathbf{n}}$ the tensorial expression in L_3 is :

$$\mathbf{T}_3 = \tfrac{1}{b}(\lambda(\vec{\mathbf{l}} \bigotimes \vec{\mathbf{n}} + \vec{\mathbf{n}} \bigotimes \vec{\mathbf{l}}) + ((\tfrac{c^2}{b^2} - \tfrac{a}{b})\lambda - \tfrac{a}{b})\vec{\mathbf{n}} \bigotimes \vec{\mathbf{n}} + \lambda\vec{\mathbf{X}}_2 \bigotimes \vec{\mathbf{X}}_2$$
$$+ (1 - \tfrac{c}{b}\lambda)(\vec{\mathbf{n}} \bigotimes \vec{\mathbf{X}}_2 + \vec{\mathbf{X}}_2 \bigotimes \vec{\mathbf{n}})$$

Choosing a suitable base we have a=c=0 and b=1 (that is $\vec{\mathbf{n}}^2 = 0$, $\vec{\mathbf{l}}^2 = 0$, $\vec{\mathbf{n}}.\vec{\mathbf{X}}_2 = 0$, $\vec{\mathbf{l}}.\vec{\mathbf{X}}_2 = 0$ and $\vec{\mathbf{X}}_2{}^2 = 1$). .

On the basis $\{\vec{\mathbf{l}}, \vec{\mathbf{n}}, \vec{\mathbf{X}}_2, \vec{\mathbf{X}}_3\}$ the stress-energy tensor would:

$$\mathbf{T} = \lambda(\vec{\mathbf{l}} \bigotimes \vec{\mathbf{n}} + \vec{\mathbf{n}} \bigotimes \vec{\mathbf{l}}) + \lambda\vec{\mathbf{X}}_2 \bigotimes \vec{\mathbf{X}}_2 + (\vec{\mathbf{n}} \bigotimes \vec{\mathbf{X}}_2 + \vec{\mathbf{X}}_2 \bigotimes \vec{\mathbf{n}}) + p_3\vec{\mathbf{X}}_3$$

It coincides with G. S. Hall and D. A. Negm [1]. However the reduction to particular values of **a,b** and **c** has to be done with caution because these values depend on the structure of the operator (or stress-energy tensor) and are not totally arbitrary. This scheme is rejected because does not fulfil the conditions weak or strong energy [5].

5 Case of irreducible minimum polynomial type Segré [$\overline{\mathrm{w}}$,w11] .

In this case the irreducible minimum polynomial corresponding to a operator M, invariant in the lorentzian subspace $\mathbf{L}_2$, has no real roots. In $\mathbf{L}_2$ this minimum polynomial is of grade 2, and his roots are conjugated complex. The minimum polynomial on $\mathbf{L}_2$ is : $P_2 \equiv ((\mathbf{M} - a\mathbf{I})^2 + b^2\mathbf{I})$.
The purpose of our study is the equation: $(\mathbf{M} - a\mathbf{I})^2 + b^2\mathbf{I} = 0$ whose roots $(\mathbf{M})$ are complex matrices. Let $\mathbf{H} = (\frac{1}{b})(\mathbf{M} - a\mathbf{I})$; hence $\mathbf{H}^2 + \mathbf{I} = (\mathbf{0})$.
It is verified

$$(\mathbf{H}^2 + \mathbf{I})\vec{\mathbf{X}} = \vec{\mathbf{0}} \tag{11}$$

for $\vec{\mathbf{X}} \in \mathbf{L}_2$.
In $\mathbf{L}_2$ for all $\vec{\mathbf{X}}$ there is an $\vec{\mathbf{Y}}$ so that $\mathbf{H}(\vec{\mathbf{X}}) = \vec{\mathbf{Y}}$. Hencefor

$$\mathbf{H}(\vec{\mathbf{X}}) = \vec{\mathbf{Y}}$$
$$\mathbf{H}(\vec{\mathbf{Y}}) = -\vec{\mathbf{X}}$$

It is necessary and sufficient condition to fulfill 11.

5.1 Some properties.

We have

$$\vec{\mathbf{X}}^2 + \vec{\mathbf{Y}}^2 = 0$$

That means that $\vec{\mathbf{X}}$ is timelike and $\vec{\mathbf{Y}}$ is spacelike (or vice versa), or $\vec{\mathbf{X}}$ and $\vec{\mathbf{Y}}$ are null vectors. We can use as a base in lorentzian space $\mathbf{L}_2$, two vectors $\vec{\mathbf{X}}$ e $\vec{\mathbf{Y}}$, in principle without determining their timelike, spacelike or nullvector nature, meanwhile we do not define a metric.

For the sake of convenience we make $\vec{\mathbf{X}} = \vec{\mathbf{n}}$ and $\vec{\mathbf{Y}} = \vec{\mathbf{l}}$. These two vectors verifie:

$$\mathbf{H}(\vec{\mathbf{l}}) = \vec{\mathbf{n}}; \mathbf{H}(\vec{\mathbf{n}}) = -\vec{\mathbf{l}};$$

Now we have the metric: $(\mathbf{G}_2) = \begin{pmatrix} 0 & w \\ w & 0 \end{pmatrix}$
where $\vec{\mathbf{l}}.\vec{\mathbf{n}} = w$ y $\vec{\mathbf{l}}^2 = 0$ y $\vec{\mathbf{n}}^2 = 0$
On that basis we have:

$$(\mathbf{H}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$(\mathbf{M}) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

5.2 Stress-energy type 4, associated.

The covariants components of the associated tensor to $\mathbf{M}$ in $\mathbf{L_2}$ are those of the matrix $\mathbf{T}_2 = \mathbf{G}_2\mathbf{M}$; that is:

$$(\mathbf{T}_2) = \mathbf{G}_2\mathbf{M} = \begin{pmatrix} wb & wa \\ wa & -wb \end{pmatrix}$$

Making $wb = \sigma_1$ and $wa = 2\sigma_0$, the stress-energy tensor respect the null base $(\vec{\mathbf{l}}, \vec{\mathbf{n}}, \vec{\mathbf{X}}_2, \vec{\mathbf{X}}_3)$ is:

$$\mathbf{T} = \sigma_1(\vec{\mathbf{l}} \otimes \vec{\mathbf{l}} - \vec{\mathbf{n}} \otimes \vec{\mathbf{n}}) + 2\sigma_0(\vec{\mathbf{l}} \otimes \vec{\mathbf{n}} + \vec{\mathbf{n}} \otimes \vec{\mathbf{l}}) + p_2\mathbf{X_2} \otimes \mathbf{X_2} + p_3\mathbf{X_3} \otimes \mathbf{X_3} \tag{12}$$

in cotravariant base $(\vec{\mathbf{l}}, \vec{\mathbf{n}}, \vec{\mathbf{X}}_2, \vec{\mathbf{X}}_3)$. It coincides with the expression presented by G. S. Hall and D. A. Negm [1]. This scheme is equally, in principle, rejected because of not to fulfil the conditions weak or strong of energy [5].

6 Conclusions

In this article we have rewritten a work on classification of selfadjoint operators. The canonical forms of the selfadjoint operators are devoloped, and the stress-energy tensors respect pseudoortonormal and orthonormal bases are gotten. Two of the 4 canonical forms are discarded because because of not to fulfil the weak or strong conditions of energy. The stress-energy tensors we are dealing with a pseudoortonormal base, coincide with the obtained by other authors ([1] y [3]) although it admit deeper interpretations.
The stress-energy tensor type 2 (the associated with the operator type Segré [2.1,1], or with the radical space of index 2), represented as an orthonormal base, contains a term of flow. It is the only that allows schemes flows as the caloric in a rest reference frame. In a limit case,this stress-energy tensor becomes the radiation stress-energy tensor in vacuum.

ANNEX

A Passage from a real null base (or semiisotrop) to a orthonormal base and vice versa in L_2

. We choose a base with the vectors $\vec{\mathbf{U}}$ and $\vec{\mathbf{X}}$ in L_2, and other base $\vec{\mathbf{l}}$ and $\vec{\mathbf{n}}$ in L_2 too fulfilling:

$$\vec{\mathbf{l}} = a_l\vec{\mathbf{X}} + b_l\vec{\mathbf{U}}$$
$$\vec{\mathbf{n}} = a_n\vec{\mathbf{X}} + b_n\vec{\mathbf{U}}$$

where
$\vec{\mathbf{l}}^2 = 0$; $\vec{\mathbf{n}}^2 = 0$; $\vec{\mathbf{l}}.\vec{\mathbf{n}} = w$
$\vec{\mathbf{X}}^2 = +1$; $\vec{\mathbf{U}}^2 = -1$; $\vec{\mathbf{X}}.\vec{\mathbf{U}} = 0$
from these conditions is clear:
$a_l = \varepsilon b_l = \alpha$
$a_n = \varepsilon b_n = \beta$
And the equations to change basis would be:

$$\vec{\mathbf{l}} = \alpha(\vec{\mathbf{X}} + \varepsilon\vec{\mathbf{U}})$$
$$\vec{\mathbf{n}} = \beta(\vec{\mathbf{X}} - \varepsilon\vec{\mathbf{U}})$$

$$\vec{\mathbf{X}} = \frac{1}{2}(\frac{\vec{\mathbf{l}}}{\alpha} + \frac{\vec{\mathbf{n}}}{\beta})$$
$$\vec{\mathbf{U}} = \frac{\varepsilon}{2}(\frac{\vec{\mathbf{l}}}{\alpha} - \frac{\vec{\mathbf{n}}}{\beta})$$

being $2\alpha\beta = w$ y $\varepsilon = \pm 1$
α y β depend on null vectors components values $\vec{\mathbf{l}}$ y $\vec{\mathbf{n}}$ respect an ortonormal reference system.

B

Base at rest respect an observer in L_2.

On this basis, the metric would have a representation matrix:

$$(\mathbf{G}'_2) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

We pass from the base ($\vec{\mathbf{n}}$,$\vec{\mathbf{l}}$) of metric $(\mathbf{G}_2) = \begin{pmatrix} 0 & w \\ w & 0 \end{pmatrix}$

(where $w = \vec{\mathbf{l}}.\vec{\mathbf{n}}$) to the base at rest **G'**$_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ The change of base is determined by the equations:

$$\vec{\mathbf{n}} = \beta(-\varepsilon\vec{\mathbf{U}} + \vec{\mathbf{X}})$$
$$\vec{\mathbf{l}} = \alpha(\varepsilon\vec{\mathbf{U}} + \vec{\mathbf{X}})$$

References

[1] **Hall G.S. (1985),International Journal of Theoretical Physics,26,**405

[2] **Lichnerovicz A. (1955),Théories relativistes de la gravitation et de l'eléctromagnetisme,Masson,Paris**

[3] **M.J. Rebouças, J. Santos y A.F.F. Teixeira,(2003)gr-qc/0312064**

[4] **José M.M. Senovilla and Raül Vera (1999),gr-qc/9902035V1**

[5] **Hawking S W, and Ellis G F R (1973),The large Structure of space-time; Cambridge University Press, Cambridge**